On Minkowski sums of simplices

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Abstract

We investigate the structure of the Minkowski sum of standard simplices in $\mathbb{R}^r$. In particular, we investigate the one-dimensional structure, the vertices, their degrees and the edges in the Minkowski sum polytope.

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1 Introduction and Definitions

Let $[r] = \{1, 2, \dots, r\}$. The *standard simplex* $\Delta_{[r]}$ of dimension $r-1$ is given by

$$\Delta_{[r]} = \{(x_1, \dots, x_r) \in \mathbb{R}^r : x_i \geq 0 \text{ for all } i\ , x_1 + \cdots + x_r = 1\}.$$

Each subset $F \subseteq [r]$ yields a *face* Δ_F of $\Delta_{[r]}$ given by

$$\Delta_F = \{(x_1, \dots, x_r) \in \Delta_{[r]} : x_i = 0 \text{ for } i \notin F\}.$$

Clearly Δ_F is itself a simplex embedded in $\mathbb{R}^r$. If $\mathcal{F}$ is a family of subsets of $[r]$, then we can form the *Minkowski sum* of simplices

$$P_{\mathcal{F}} = \sum_{F \in \mathcal{F}} \Delta_F = \left\{ \sum_{F \in \mathcal{F}} x_F : x_F \in \Delta_F \text{ for each } F \in \mathcal{F} \right\}.$$

If $|F| = 2$ for all $F \in \mathcal{F}$, then the polytope $P_{\mathcal{F}}$ is called a *graphical zonotope*. Graphical zonotopes were studied by West et. al. [4], [11], but several questions about them have gone unanswered. Minkowski sums of simplices have more recently been studied by Feichtner and Sturmfels [3], and by Postnikov [9]. These later papers focus on a the case when the collection $\mathcal{F}$ is a *building set*, i.e. $\mathcal{F}$ contains all singletons, and has the property that, for any $F_1, F_2 \in \mathcal{F}$, $F_1 \cap F_2 \neq \emptyset$ implies that $F_1 \cup F_2 \in \mathcal{F}$. This property implies that the polytope $P_{\mathcal{F}}$ is simple. Applications of Minkowski sums of simplices appear in the paper of Morton et. al. [8]. Minkowski sums of simplices have also appeared in the work of Conca [2] and of Herzog and Hibi [6], under the name transversal polymatroids.

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Observation 1.1 *The dimension of the polytope $P_{\mathcal{F}}$ is given by* $\dim(P_{\mathcal{F}}) = n - c$ *where*

$$n = \left| \bigcup_{F \in \mathcal{F}} F \right| \in [r]$$

and c is the number of connected components of $\Delta_{\mathcal{F}}$, the simplicial complex with facets $\max(\mathcal{F})$.

Proof. For each $F \in \mathcal{F}$ present Δ_F by $\Delta_F = \{(x_{F;1}, \ldots, x_{F;n}) \in \Delta_{[r]} : x_{F;i} = 0 \text{ for } i \notin F\}$. We then use the equation $\sum_{i \in F} x_{F;i} = 1$ for each F to obtain $x_{F;\max(F)} = 1 - \sum_{i \in F \setminus \{\max(F)\}} x_{F;i}$ and eliminate $x_{F;\max(F)}$ in the Minkowski sum. By then counting the free variables, we have the observation. $\square$

From the following more graph theoretic point of view we also can consider the following: Let $\Delta_1(\mathcal{F})$ be the 1-dimensional skeleton of $P_{\mathcal{F}}$.

Observation 1.2 *The dimension of the polytope $P_{\mathcal{F}}$ is given by* $\dim(_{\mathcal{F}}) = |E(T_{\mathcal{F}})|$, *the number of edges in a spanning forest of* $\Delta_1(\mathcal{F})$.

A *face* of $P_{\mathcal{F}}$ is a subset of $P_{\mathcal{F}}$ on which a linear function is maximized. A vector $c = (c_1, \ldots, c_r) \in \mathbb{R}^r$ defines a partition $C = (C_1, C_2, \ldots, C_s)$ of $[r]$ into nonempty subsets, so that $c_{i_1} = c_{i_2}$ when i_1 and i_2 are in the same part of the partition, and $c_{i_1} < c_{i_2}$ whenever $i_1 \in C_{\ell_1}, i_2 \in C_{\ell_2}, \ell_1 < \ell_2$. Then the points of the face Q that maximizes $c^T x$ satisfy the equations

$$\sum_{i \in C_\ell} x_i = |\{F \in \mathcal{F} : F \cap C_\ell \neq \emptyset, F \cap C_m = \emptyset \text{ for } m > \ell\}|.$$

for $\ell = 1, 2, \ldots, s$. The face that maximizes $c^T x$ is therefore the Minkowski sum of the simplices in the family

$$\mathcal{F}^C := \{F \cap C_{\ell_F} : F \in \mathcal{F}, F \cap C_{\ell_F} \neq \emptyset, F \cap C_m = \emptyset \text{ for } m > \ell_F\}$$

The dimension of the face is determined by the number of connected components of the simplicial complex $\Delta_{\mathcal{F}^C}$. If $\Delta_{\mathcal{F}^C}$ is obtained from $\Delta_{\mathcal{F}}$ by splitting one of the components of $\Delta_{\mathcal{F}}$ in two, then the corresponding face of $P_{\mathcal{F}}$ is a facet, and the the coefficients of the vector c corresponding to C can be assumed to be 0 and 1. Therefore, all facets of $P_{\mathcal{F}}$ are of the form $\sum_{i \in D} x_i = t$ for some subset D of $[r]$ and integer t. When $\Delta_{\mathcal{F}^C}$ has exactly one component of size two, say $\{i, j\}$, and otherwise all isolated elements, then the corresponding face of $P_{\mathcal{F}}$ is an edge parallel to $e_i - e_j$. Vertices of $P_{\mathcal{F}}$ are points that maximize linear functions $c^T x$ in which all components of c are distinct. If $c_1 < c_2 < \cdots < c_r$ then component v_i of the vertex that maximizes $c^T x$ equals the number of sets F for which i is the largest element. In particular, vertices of $P_{\mathcal{F}}$ have integer coordinates.

2 Minkowski sum of a fixed number of simplices

Suppose that $\mathcal{F}$ consists of k subsets $F_1, F_2, \ldots, F_k$ of $[r]$. For each $i \in [r]$, define $N_{\mathcal{F}}(i) = \{j \in [k] : i \in F_j\}$. Let A be a subset of $[r]$ so that $N_{\mathcal{F}}(i_1) = N_{\mathcal{F}}(i_2)$ whenever i_1 and i_2 are in A. We would like to show how the combinatorial type of $P_{\mathcal{F}}$ can be inferred from that of $P_{\mathcal{F}'}$, where $\mathcal{F}'$ is

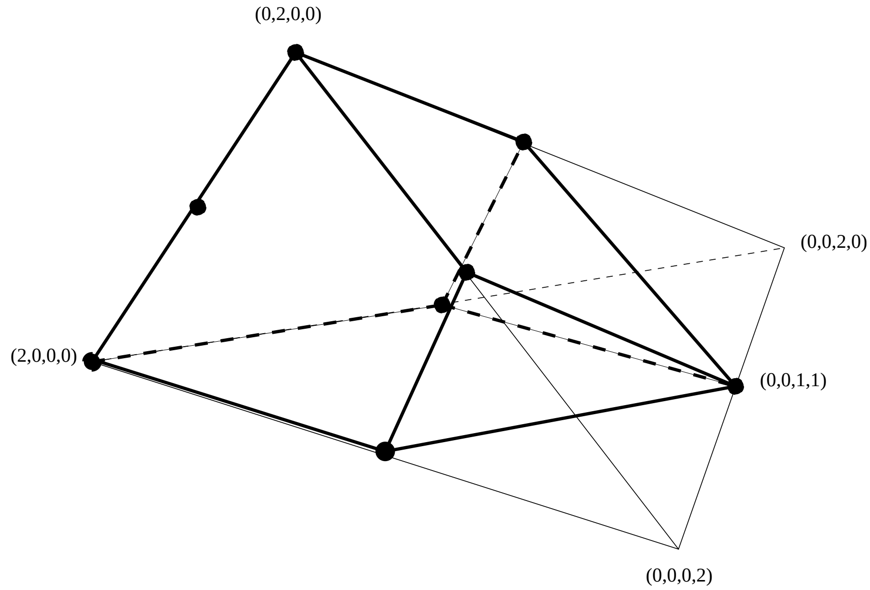

Figure 1: A sum of two triangles

obtained from $\mathcal{F}$ by replacing each appearance of A in a set F by the one-element set $m = \max(A)$. Afterward, we will restrict our attention to families in which all of the $N_{\mathcal{F}}(i)$ are distinct.

Every point $y \in P_{\mathcal{F}'}$ corresponds to the simplex $\Delta(y) := \{z \in \mathbb{R}^r : z_i = y_i, i \notin A, \sum_{i \in A} z_i = y_m, z_i \geq 0, i \in A\}$ contained in $P_{\mathcal{F}}$. Note that $\Delta(y)$ is $(|A|-1)$-dimensional if $y_m > 0$ and a point otherwise. Let $\mathcal{F}''$ be the face of $\mathcal{F}$ where $y_m = 0$. The combinatorial type of $P_{\mathcal{F}}$ is therefore that of $\Delta_A \times P_{\mathcal{F}'}$, with (if $P_{\mathcal{F}''}$ is nonempty) the face $\Delta_A \times P_{\mathcal{F}''}$ collapsed to a copy of $P_{\mathcal{F}''}$. In the case that $|A| = 2$, $P_{\mathcal{F}}$ is a wedge over $P_{\mathcal{F}'}$ with foot $P_{\mathcal{F}''}$.

EXAMPLE Consider the family $\mathcal{F} = \{\{1,2,3\},\{1,2,4\}\}$ of subsets of [4]. Then $N_{\mathcal{F}}(i) = \{1,2\}$ for all i in $A = \{1,2\}$. The polytope $P_{\mathcal{F}}$ is drawn in Figure 1. The polytope $P_{\mathcal{F}'}$ is the two-dimensional cube that is the top face of the drawing. $P_{\mathcal{F}''}$ is the vertex $(0,0,1,1)$.

Proposition 2.1 *Every vertex of $P_{\mathcal{F}}$ is of the form $y + y_m(e_i - e_m)$, where y is a vertex of $P_{\mathcal{F}'}$ and $i \in \{1,2,\ldots,m\}$. Vertices $y + y_m(e_i - e_m)$ and $y' + y'_m(e_j - e_m)$ of $P_{\mathcal{F}}$ are adjacent in $P_{\mathcal{F}}$ if*

1. *$y = y'$ and $y_m > 0$ or*

2. *y is adjacent to y' in $P_{\mathcal{F}'}$ and either $i = j$ or $y_m y'_m = 0$.*

Note that no vertex has more than one component of A nonzero, because the vertices of the simplex Δ_A have only one nonzero component.

We consider first the case in which $\mathcal{F}$ consists of two sets, F and F'. In the special case where each of the sets $F \setminus F'$, $F \cap F'$ and $F' \setminus F$ has exactly one element, say 1,2 and 3 respectively, then $F = \{1,2\}$ and $F' = \{2,3\}$ and the Minkowski sum $P = \Delta_F + \Delta_{F'}$ is the convex hull of $(1,1,0), (0,2,0), (0,1,1)$ and $(1,0,1)$ in $\mathbb{R}^3$, which constitutes a two-dimensional rhombus within the positive octant of the plane $x + y + z = 2$.

We now argue that the generic Minkowski sum of two simplices roughly has the structure of such a rhombus, if each of $F \setminus F', F \cap F'$, and $F' \setminus F$ is nonempty.

By assigning the 1st, 2nd and 3d coordinate axis of $\mathbb{R}^3$ to these parts respectively, we can partition the vertices of $P = \Delta_F + \Delta_{F'}$ in the following "rhombus"-way: A vertex $e_i + e_j$ of $P_{\mathcal{F}}$ is of type $A = (1, 1, 0)$ if $i \in F \setminus F'$ and $j \in F \cap F'$, of type $B = (0, 2, 0)$ if $i = j \in F \cap F'$, of type $C = (0, 1, 1)$ if $i \in F \cap F'$ and $j \in F' \setminus F$ and of type $D = (1, 0, 1)$ if $i \in F \setminus F'$ and $j \in F' \setminus F$. Note that the rhombus formed by A, B, C and D in $\mathbb{R}^3$ has edges AB, BC, CD and DA. With this setup we have the following.

Lemma 2.2 *If A, B, C and D are the points in $\mathbb{R}^3$ as here above and $F \setminus F'$, $F \cap F'$ and $F' \setminus F$ are all nonempty, then there are no AC nor BD type edges of $P = \Delta_F + \Delta_{F'}$.*

Proof. The original rhombus does not have AC or BD edges. $\square$

By the above Lemma 2.2 we have the following corollary that describes the structure of a Minkowski sum of two standard simplices to be roughly that of the rhombus mentioned above.

Corollary 2.3 *If $F, F \subseteq [r]$ then the edges, or one-dimensional faces, of $P = \Delta_F + \Delta_{F'}$ are of the following types:*

1. *Internal XX edges, where both the endvertices are of type $X \in \{A, B, C, D\}$.*
2. *XY edges, with $XY \in \{AB, BC, CD, DA\}$, where one endvertex is of type X and the other of type Y.*

Theorem 2.4 *Let $F, F' \subseteq [r]$ and let u be a vertex of the polytope $P_{\mathcal{F}}$.*

1. *If u is of type A, B or C, then $\deg(u) = |F \cup F'| - 1$.*
2. *If u is of type D, then $\deg(u) = |F| + |F'| - 2$.*

Proof. If u is of type B, say $u = 2e_i$, then u is adjacent to all $|F \cap F'| - 1$ other vertices of type B, and all type A and C vertices of the form $e_i + e_j$, where $j \in (F \setminus F') \cup (F' \setminus F)$. If u is of type A, say $u = e_i + e_j$, with $i \in F \setminus F'$ and $j \in F \cap F'$, then u is adjacent to two kinds of type A vertices: $|F \cap F'| - 1$ vertices $e_i + e_k$ with $k \in (F \cap F') \setminus \{j\}$ and $|F \setminus F'| - 1$ vertices $e_k + e_j$ with $k \in F \setminus (F' \cap \{i\})$. Also, u is adjacent to $|F' \setminus F|$ type D vertices $e_i + e_k$ with $k \in F' \setminus F$, and finally u is adjacent to the vertex $2e_j$. If u is of type D, say $u = e_i + e_j$ with $i \in F \setminus F'$ and $j \in F' \setminus F$, then u is adjacent to $|(F \setminus F') \cup (F \setminus F')| - 2$ vertices of type D obtained by replacing either e_i or e_j by an e_k for $k \in (F \setminus F') \cup (F \setminus F')$, and u is adjacent to $|F \cap F'|$ vertices of each type A and C, obtained by replacing e_i or e_j by an e_k for $k \in F \cap F'$. $\square$

Corollary 2.5 *Let $F, F' \subseteq [r]$ and $P = \Delta_F + \Delta_{F'}$.*

1. *The total number of vertices of P is $|F| \cdot |F'| - |F \cap F'|(|F \cap F'| - 1)$.*
2. *The total number of one-dimensional faces (edges) of $P = \Delta_F + \Delta_{F'}$ is given by*

$$\frac{1}{2}\left[|F \setminus F'| \cdot |F' \setminus F|(|F| + |F'| - 2) + |F \cap F'|(|F \cup F'| - 1)(|F \setminus F'| + |F' \setminus F| + 1)\right].$$

Proof. The number of vertices of degree $|F| + |F'| - 2$ in P is $|F \setminus F'| \cdot |F' \setminus F|$. By Theorem 2.4 the remaining vertices of P all have degree $|F \cup F'| - 1$. By the Hand-Shaking Theorem the total number of edges, or one-dimensional faces, is given as stated. $\square$

Assuming that $F \cup F' = [r]$, then the maximum value of $|F| + |F'| - 2$ (provided $F \setminus F'$ and $F' \setminus F$ are nonempty) is $2r - 4$, which occurs when $F = [r-1]$ and $F' = [r] \setminus \{1\}$. Considering the distribution of the two possible degrees of a Minkowski sum of two simplices $P = \Delta_F + \Delta_{F'}$, we have the following.

Proposition 2.6 *Let $r \in \mathbb{N}$ be fixed. If $F, F' \subseteq [r]$ and $P = \Delta_F + \Delta_{F'}$ is of dimension $r-1$, then the average degree $\overline{\deg}(P)$ satisfies*

$$r - 1 \leq \overline{\deg}(P) < \frac{10}{9}(r-1).$$

Moreover, the lower bound is attained iff (i) $F \subseteq F'$, (ii) $F' \subseteq F$ or (iii) $|F \cap F'| = 1$. Also, $\overline{\deg}(P)/(r-1)$ can become arbitrarily close to $10/9$ for large r.

Proof. We introduce the variables x, y and z by $x = |F \setminus F'|$, $y = |F' \setminus F|$ and $z = |F \cap F'|$. Here we have the boundary condition $x, y \geq 0$ and $x + y + z = r$, and since P is assumed to have dimension $r-1$ we have $z \geq 1$ or $0 \leq x + y \leq r - 1$. By Corollary 2.5 and the Hand-Shaking Theorem we obtain that

$$\begin{aligned}
\overline{\deg}(P) &= 2\frac{|E(\Delta_1(\mathcal{F}))|}{|V(\Delta_1(\mathcal{F}))|} \\
&= \frac{|F \setminus F'| \cdot |F' \setminus F|(|F| + |F'| - 2) + |F \cap F'|(|F \cup F'| - 1)(|F \setminus F'| + |F' \setminus F| + 1)}{|F| \cdot |F'| - |F \cap F'|(|F \cap F'| - 1)} \\
&= \frac{xy(2r - 2 - x - y) + (r-1)(r - x - y)(x + y + 1)}{(r-y)(r-x) - (r-x-y)(r-x-y-1)}.
\end{aligned}$$

As a function of x and y we note that $\overline{\deg}(P) = \overline{\deg}(x, y)$ is symmetric, has the value of $r-1$ on the boundary of the triangle bounded by $x = 0$, $y = 0$ and $x + y = r - 1$. By Theorem 2.4 the value $\overline{\deg}(x, y)$ is strictly larger than $r - 1$ inside the triangle. If the maximum value of $\overline{\deg}(x, y)$ is $\overline{\deg}_{\max}(r)$, then $(10r - 13)/9 < \overline{\deg}_{\max}(r) < 10(r-1)/9$, but $\overline{\deg}_{\max}(r) - (10r - 13)/9$ tends to zero when r tends to infinity. $\square$

Remark: In fact, for any $\epsilon > 0$ there is an r_0 such that for any $r \geq r_0$ we have

$$r - 1 \leq \overline{\deg}(P) < \frac{10r - 13}{9} + \epsilon.$$

The *f-polynomial* $f_P(q)$ of a d-dimensional polytope P is $\sum_{i=0}^{d} f_i q^i$, where f_i is the number of i-dimensional faces of P. Postnikov [9] shows that $f_{P \times Q}(q) = f_P(q) f_Q(q)$ and gives an elegant formula for $f_{P_{\mathcal{F}}}(q)$ in the case that $\mathcal{F}$ is a building set. If we assume that A, $\mathcal{F}'$ and $\mathcal{F}''$ are as in the discussion preceding Proposition 2.1, the f-polynomial can be decomposed as follows:

Proposition 2.7 $f_{P_{\mathcal{F}}}(q) = f_{\Delta_A}(q) f_{P_{\mathcal{F}'}}(q) - f_{\Delta_A}(q) f_{P_{\mathcal{F}''}}(q) + f_{P_{\mathcal{F}''}}(q)$.

In the Example, $f_{P_{\mathcal{F}}}(q) = 7 + 11q + 6q^2 + q^3 = (2+q)(4 + 4q + q^2) - (2+q)(1) + 1$.

If $P_{\mathcal{F}}$ is the sum of two simplices Δ_F and $\Delta_{F'}$, then one can easily check that $P_{\mathcal{F}} = \Delta_F \times \Delta_{F'}$ when $|F \cap F'|$ is 0 or 1. This allows us to describe the f-polynomials of sums of two simplices quite easily, using the proposition with $A = F \cap F'$.

Corollary 2.8 *If $\mathcal{F} = \{F, F'\}$, where $F \cap F' = \{1, 2, \dots, m\}$, then*

$$f_{P_{\mathcal{F}}}(q) = f_{\Delta_{F\cap F'}}(q) f_{\Delta_{(F\cup m)}\times\Delta_{(F'\cup m)}}(q) - f_{\Delta_{F\cap F'}}(q) f_{\Delta_F\times\Delta_{F'}}(q) + f_{\Delta_F\times\Delta_{F'}}(q)$$

In particular, the number of vertices of $P_{\mathcal{F}}$ is $|F\cap F'|(|F\backslash F'|+1)(|F'\backslash F|+1) - |F\cap F'||F\backslash F'||F'\backslash F| + |F\backslash F'||F'\backslash F| = |F\cap F'|(|F\backslash F'|+|F'\backslash F|+1) + |F\backslash F'||F'\backslash F|$ which is consistent with Corollary 2.5.

We will now generalize the results that we obtained for the sum of two simplices to larger sums.

Definition 2.9 *For $k \in \mathbb{N}$ let $\mathcal{H}(k)$ be the family of k subsets of $[2^k-1]$ so that for $i = 1, 2, \dots, 2^k-1$, $N_{\mathcal{H}(k)}(i)$ is the i^{th} (in lexicographic order) nonempty subset of $[k]$. Then $P(k) := P_{\mathcal{H}(k)}$ is called the k^{th}* master polytope.

Definition 2.10 *Let $\mathcal{F} = (F_1, \dots, F_k)$ and let u be a point in $P_{\mathcal{F}}$. Then $h_{\mathcal{F}}(u)$ is the point v in $P(k)$ for which, for $i = 1, 2, \dots, 2^k-1$, we set*

$$v_i = \begin{cases} \sum_{j:N_{\mathcal{F}}(j)=N_{\mathcal{H}(k)}(i)} u_j & \text{if there is a } j \text{ with } N_{\mathcal{F}}(j) = N_{\mathcal{H}(k)}(i), \\ 0 & \text{otherwise} \end{cases}$$

Remark: Another way to look at $v = h_{\mathcal{F}}(u)$ is as follows: For $\mathcal{F} = (F_1, \dots, F_k)$ let u be a point in $P_{\mathcal{F}}$ for which $u_iu_j > 0$ implies $N_{\mathcal{F}}(j) \neq N_{\mathcal{F}}(i)$. Then let $h_{\mathcal{F}}(u)$ be the point v in $P(k)$ where $v_{\ell_i} = u_i$ where ℓ_i is the unique element in $[2^k-1]$ with $N_{\mathcal{H}(k)}(\ell_i) = N_{\mathcal{F}}(i)$ for each $i \in [r]$.

Theorem 2.11 *For $\mathcal{F} = (F_1, \dots, F_k)$ the point $u \in P_{\mathcal{F}}$ is a vertex of $P_{\mathcal{F}}$ if, and only if, the following conditions are met.*

1. *Each instance of $u_{i_\alpha}u_{i_\alpha} > 0$, $N_{\mathcal{F}}(i_\alpha) = N_{\mathcal{F}}(i_\beta)$ implies that $i_\alpha = i_\beta$.*
2. *$h_{\mathcal{F}}(u)$ is a vertex of the polytope $P(k)$.*

Proof. (Theorem 2.11 Sketch) For a point $u = e_{i_1} + \cdots + e_{i_k}$ of $P_{\mathcal{F}}$ we first note that if $N_{\mathcal{F}}(i_\alpha) = N_{\mathcal{F}}(i_\beta)$ and $i_\alpha \neq i_\beta$, then $u = (v+w)/2$ where v and w are the points of $P_{\mathcal{F}}$ obtained from u on one hand by replacing i_α by i_β to get v and on the other hand by replacing i_β by i_α to get w. Hence, the first condition is necessary.

Assume that u satisfies the first condition and that $h_{\mathcal{F}}(u)$ is an extreme point of $P(k)$. Since there is a supporting hyperplane in $\mathbb{R}^{2^k-1}$ containing $h_{\mathcal{F}}(u)$ there is a corresponding supporting hyperplane in $\mathbb{R}^n$ containing u, showing that u is a vertex of $P_{\mathcal{F}}$.

Assume finally that u satisfies the first condition and that $h_{\mathcal{F}}(u)$ is not an extreme point of $P(k)$. In this case $h_{\mathcal{F}}(u)$ is a proper convex combination of extreme points of $P(k)$. Since the first condition is satisfied, there are corresponding points of $P_{\mathcal{F}}$, such that u is a proper (in fact the same!) convex combination of these. This completes the proof. □

For $\mathcal{F} = (F_1, \dots, F_k)$ let $A_1, \dots, A_h$ be the vertices of the polytope $P(k)$. Similar to the case when $k = 2$ we have the following.

Theorem 2.12 *If $\mathcal{F} = (F_1, \dots, F_k)$, then the edges, or one-dimensional faces, of $P_{\mathcal{F}}$ are of the following types:*

1. *Internal A_iA_i type edges, where both the endvertices are of type A_i for some $i \in \{1, \dots, m\}$.*

2. A_iA_j *type edges, where* A_iA_j *is an edge of the master polytope* $P(k)$.

Proof. (Sketch) Similarly to the proof of Lemma 2.2 (although with a bit more elaborate indexing scheme) one can show that there is a supporting hyperplane in $\mathbb{R}^n$ of P containing the vertex of type A_i and the vertex of type A_j if, and only if, there is a corresponding supporting hyperplane in $\mathbb{R}^{2^k-1}$ of $P(k)$ containing the vertices A_i and A_j. □

Theorems 2.11 and 2.12 both reduce the structure of $P_{\mathcal{F}} \subseteq \mathbb{R}^n$ to considerations of the master polytope $P(k) \subseteq \mathbb{R}^{2^k-1}$.

3 Function Representation of Integer Points of $P_{\mathcal{F}}$

As in the previous section, we assume that $\mathcal{F} = (F_1, \ldots, F_k)$, an ordered collection of k subsets of $[r]$. A function $f : [k] \to [r]$ that satisfies $f(i) \in F_i$ for each i will be called a *rep-function*. For a rep-function f we define $u(f) := e_{f(1)} + \cdots + e_{f(k)}$.

Claim 3.1 *For functions* $f, g : [k] \to [m]$ *we have*

1. $u(f) + u(g) = u(\min\{f, g\}) + u(\max\{f, g\})$.
2. *If* $f \neq g$*, then* $u(f) \neq u(\min\{f, g\})$.

In the case $u(f) = u(g)$, we obtain by Claim 3.1 that $u(f) = u(g) = (u(\min\{f, g\}) + u(\max\{f, g\}))/2$. Hence, if an integer point $u \in P_{\mathcal{F}}$ can be represented by two distinct functions f and g, then it is not a vertex of the type polytope $P(k)$. The interesting part is the converse.

Lemma 3.2 *If* v *is an integer point in* $P_{\mathcal{F}}$ *that is not a vertex of* $P_{\mathcal{F}}$*, and an edge of the smallest face containing* v *is parallel to* $e_{i_1} - e_{i_2}$*, then* $P_{\mathcal{F}}$ *contains the points* $v + e_{i_1} - e_{i_2}$ *and* $v - e_{i_1} + e_{i_2}$.

Proof. First note that $v_{i_1} \neq 0$ and $v_{i_2} \neq 0$, because otherwise all points on the smallest face containing v would satisfy $x_{i_1} = 0$ or $x_{i_2} = 0$, contradicting the assumption that there is an edge of this face parallel to $e_{i_1} - e_{i_2}$. If v is on a facet of $P_{\mathcal{F}}$ given by $\sum_{i \in T} x_i = t$ for some $T \subset [r]$ and integer t, then this equation is satisfied by all points in the smallest face containing v. That means that i_1 and i_2 are either both in or both outside of T. Thus $v + e_{i_1} - e_{i_2}$ and $v - e_{i_1} + e_{i_2}$ will satisfy any equations that v satisfies. Furthermore, any inequality $x_i \geq 0$ or $\sum_{i \in T} x_i \leq t$ that v satisfies strictly will also be satisfied by $v + e_{i_1} - e_{i_2}$ and $v - e_{i_1} + e_{i_2}$, because only one component is increased by 1 and one component is decreased by 1. □

Lemma 3.3 *If* f *and* g *are rep-functions and* $u(g) = u(f) + te_{i_1} - te_{i_2}$ *for* $i \neq j$ *in* $[r]$*, then there exist rep-functions* $f_1, f_2, \ldots f_{t-1}$ *so that* $u(f) + le_{i_1} - le_{i_2} = u(f_l)$ *for* $l = 1, 2, \ldots, t-1$.

Proof. Define $G_{\mathcal{F}}$ to be the bipartite graph with vertex set $\{w_j : j \in [k]\} \cup \{v_t : i \in [r]\}$ and edges $\{(w_j, v_i)\}$ for all (i, j) with $i \in F_j$. For any rep-function h, let M_h be the set of edges (w_j, v_i) for which $h(j) = i$. For every $i \in [r]\{i_1, i_2\}$, the number of edges of M_g meeting v_i equals the number of edges of M_f meeting v_i. For every $j \in [k]$, w_j is met by exactly one edge from each of M_f and M_g. On the other hand, v_{i_1} is adjacent to t more edges of M_g than M_f, and v_{i_2} is adjacent to t more edges of M_f than M_g. There therefore exists a path P from v_{i_2} to v_{i_1} that alternates between edges of M_f and M_g. Let M^1 be the set of edges obtained from M_f by replacing the edges of M_f in the path by the edges of M_g in the path. Then, for $j = 1, 2, \ldots, k$, define $f_1(j) = i$, where (w_j, v_i) is an edge of M^1. Then $u(f_1) = u(f) + e_{i_1} + e_{i_2}$. We can continue this way to get $u(f_2), \ldots, u(f_{t-1})$. □

Proposition 3.4 *Every integer point v in $P_{\mathcal{F}}$ is $u(f)$ for some rep-function f.*

Proof. The proof is by induction on the dimension of the smallest face containing v. From the first section, we know that the statement is true if true if v is a vertex. Suppose v is not a vertex. Suppose that there is an edge of the smallest face containing v that is parallel to $e_{i_1} - e_{i_2}$. Then lemma 3.2 allows us to build a segment parallel to $e_{i_1} - e_{i_2}$, containing v in its interior, and with endpoints on faces of $P_{\mathcal{F}}$ that are of lower dimension than the one containing v. By induction, the endpoints of the interval are $u(f)$ and $u(g)$ for some rep-functions f and g. Lemma 3.3 then gives us a rep-function for v. □

Theorem 3.5 *An integer point v in $P_{\mathcal{F}}$ is a vertex of $P_{\mathcal{F}}$ if and only if there is a unique rep-function f so that $u(f) = v$.*

Proof. Let v be an integer point in $P_{\mathcal{F}}$ that is not a vertex of $P_{\mathcal{F}}$. By Lemma 3.2 there are i_1 and i_2 in $[r]$ so that $P_{\mathcal{F}}$ contains the points $v - e_{i_1} + e_{i_2}$ and $v - e_{i_1} + e_{i_2}$. Let$f$ and g be the rep-functions guaranteed by Proposition 3.4 for $v - e_{i_1} + e_{i_2}$ and $v - e_{i_1} + e_{i_2}$, respectively. Let $G_{\mathcal{F}}, M_f$ and M_g be as in the proof of Lemma 3.3. Then There are two edges of M_f adjacent to v_{i_2} that are not in M_g. Therefore we can use these edges as initial edges in two different paths from v_{i_2} to v_{i_1} that alternate between edges of M_f and M_g. Swapping edges of M_f for edges of M_g along each of these alternating paths leads to two different rep-functions for v. □

The number of rep-functions for a given $\mathcal{F}$ is easy to count, it is $\Pi_{F\in\mathcal{F}}|F|$. By listing the rep-functions and the corresponding integer points $u(f)$, and striking out the $u(f)$ that appear more than once, one can list the vertices of $P_{\mathcal{F}}$. This was done by Bernd Sturmfels [1] for the polytopes $P(k)$, $k = 3, 4, 5$. He found that $P(3)$ had 41 vertices, $P(4)$ had 1015 vertices, and $P(5)$ had 59072 vertices.

4 Max-degree as function of parameters alone

In this section we determine the function $d : \mathbb{N} \to \mathbb{N}$ given by

$$d(r) = \max_{\mathcal{F}} \{\deg_{\max}(P_{\mathcal{F}})\},$$

where the maximum is taken over all multi-subsets $(F_1, \ldots, F_k)$ of $\mathbb{P}([r])$, where $k \in \mathbb{N}$ can be any integer but r is fixed. Moreover, for each fixed $k \in \mathbb{N}$ we determined the function $d_k : \mathbb{N} \to \mathbb{N}$ defined by

$$d_k(r) = \max_{|\mathcal{F}|\leq k} \{\deg_{\max}(P_{\mathcal{F}})\},$$

where the maximum is here taken over all multi-subsets $(F_1, \ldots, F_k)$ of $\mathbb{P}([r])$ where both k and r are fixed. Clearly $d(r) = \max_{k\in\mathbb{N}}\{d_k(r)\}$.

We start with the following lower bound for $d_k(r)$ and $d(r)$.

Lemma 4.1 *For $k, r \in \mathbb{N}$ we have $d_k(r) \geq k(r-k)$, and therefore $d(r) \geq \lfloor r^2/4 \rfloor$.*

Proof. Let $k \in [r]$ and let for each $i \in [k]$ let $F_i = \{i, k+1, k+2, \ldots, r\}$. Then the vertex $v = e_1 + e_2 + \cdots e_k$ is adjacent to each of the vertices $v + (e_{i_1} - e_{i_2})$, for $1 \leq i_2 \leq i_1$ and $k+1 \leq i_1 \leq r$. Therefore $d_k(r) \geq k(r-k)$, so we have in particular that $d(r) \geq \lfloor r/2 \rfloor \lceil r/2 \rceil = \lfloor r^2/4 \rfloor$. □

Another polytope that has vertices of degree $\lfloor r^2/4 \rfloor$ is the graphical zonotope for the complete bipartite graph with $\lfloor r/2 \rfloor$ vertices on one side of the bipartition and $\lceil r/2 \rceil$ vertices on the other side. West [11] proved that the graphical zonotope for the complete bipartite graph has vertices of degree ℓ for all $r-1 \leq \ell \leq \lfloor r^2/4 \rfloor$. On the other hand, every vertex of the polytope of lemma 4.1 other than v has degree $r-1$.

For a fixed vertex u, each edge of P incident to u can be identified with a multiple of a difference $e_i - e_j$ of some pair of unit vectors, where $i, j \in [r]$ are distinct. Since the collection $\{\alpha(e_i - e_j) : \alpha \in \mathbb{N}\}$ is a set of parallel vectors, at most one multiple of $e_i - e_j$ can possibly correspond to an edge incident to u. From this alone we see that the maximum number of edges incident to u is at most $\binom{r}{2}$. However, more can be said:

For a vertex u of P, let $\vec{G}(u)$ be the directed graph with the vertexset $V(\vec{G}(u)) = [r]$ where a directed edge (i, j) is present iff $u + \alpha(e_i - e_j)$ is a neighbor of u in P for some $\alpha \in \mathbb{N}$.

Proposition 4.2 *For $r \in \mathbb{N}$ and $\mathcal{F} = (F_1, \ldots, F_k) \subseteq \mathbb{P}([r])$, the digraph $\vec{G}(u)$ is acyclic and its underlying graph $G(u)$ is simple and triangle-free.*

Proof. Assume there is a cycle $(i_1, i_2, \ldots, i_h)$ in $\vec{G}(u)$. Then $u, v_1, \ldots v_h$ are all vertices of P, where $v_\ell = u + \alpha_\ell(e_{i_\ell} - e_{i_{\ell+1}})$ (here we compute cyclically, so $e_{i_{h+1}} = e_{i_1}$). This is however impossible since

$$\sum_{\ell=1}^{h} \frac{1}{\alpha_\ell} (v_\ell - u) = 0,$$

which means that there is no hyperplane containing u alone and having all the v_ℓ's strictly on one side of it. In particular for $h = 2$, there are no directed 2-cycles and hence the underlying graph $G(u)$ is simple. Also for $h = 3$, there are no directed triangles in $\vec{G}(u)$ either.

Assume now that $G(u)$ has a triangle, which then does not correspond to a directed triangle in $\vec{G}(u)$, say $v = u + \alpha(e_i - e_j)$, $v' = u + \beta(e_j - e_l)$ and $v'' = u + \gamma(e_i - e_l)$. In this case we have

$$v'' - u = \frac{\gamma}{\alpha}(v - u) + \frac{\gamma}{\beta}(v' - u),$$

which means that the vector $v'' - u$ is in the cone spanned by $v - u$ and $v' - u$. This contradicts the fact that uv'' is an edge of P. Hence, the underlying graph $G(u)$ of $\vec{G}(u)$ has no triangles. $\square$

Theorem 4.3 *For $r \in \mathbb{N}$ we have $d(r) \leq \lfloor r^2/4 \rfloor$.*

Proof. The maximum degree of a vertex u of P is by Proposition 4.2 the maximum number of edges the simple triangle free graph $G(u)$ can have. By a theorem by Mantel [7] (as a special case of Turán's Theorem [10]), the maximum number of edges of a simple triangle-free graph on r vertices is $\lfloor r^2/4 \rfloor$, hence the theorem. $\square$

By Lemma 4.1 and Theorem 4.3 we have the following corollary.

Corollary 4.4 *For $r \in \mathbb{N}$ we have $d(r) = \lfloor r^2/4 \rfloor$.*

We now turn our attention to the computation of $d_k(r)$. Note that the Minkowski sum $P_{\mathcal{F}}$ provided in the proof of Lemma 4.1 that attains the overall maximum degree $d(r)$ has $k = |\mathcal{F}| = \lfloor r/2 \rfloor$. Therefore when computing $d_k(r)$ we can assume $1 \leq k \leq r/2$.

First we need a variation of the theorem by Mantel [7]: Let G be a simple graph on n vertices and let $1 \leq k \leq n/2$.

Call G a *k-triangle-free graph*, or a *k-tr* for short, if G is triangle free and G has a vertex cover of cardinality at most k.

Theorem 4.5 *Let $n \in \mathbb{N}$ and $1 \leq k \leq n/2$. If $e_k(n)$ is the maximum number of edges of a k-tr graph G, then $e_k(n) = k(n-k)$. Moreover, if G is a k-tr graph on n vertices with $e_k(n)$ edges, then G is a complete bipartite with parts of cardinalities k and $n-k$.*

Proof. For $n \in \{1, 2\}$ the theorem is trivial. We proceed by induction and assume we have a k-tr graph on $n > 2$ vertices with the maximum number $e_k(n)$ of edges. Let $uv \in E(G)$ be an edge and since either u or v is in the vertex cover U of size k, we assume it to be u. Since G is triangle-free the set of neighbors $N(u)$ and $N(v)$ are disjoint. Let $G' = G - \{u, v\}$ be the simple graph obtained from G by removing the vertices u and v from G. By the disjointness of $N(u)$ and $N(v)$ we have $|E(G)| = |E(G')| + d(u) + d(v) - 1$.

Assume first that $v \in U$. In this case G' is a $(k-2)$-tr graph on $n-2$ vertices and hence by induction hypothesis we have $|E(G)| = |E(G')|+d(u)+d(v)-1 \leq (k-2)[(n-2)-(k-2)]+n-1 < k(n-k)$.

Now assume that $v \notin U$. In this case G' is a $(k-1)$-tr graph on $n-2$ vertices and hence by induction hypothesis we have $|E(G)| = |E(G')|+d(u)+d(v)-1 \leq (k-1)[(n-2)-(k-1)]+n-1 = k(n-k)$. Also by inducting hypothesis, $|E(G)| = k(n-k)$ can hold iff G' is a complete bipartite graph with parts of cardinalities $k-1$ and $n-k-1$, and $d(u)+d(v) = n$ (i.e. $N(u) \cup N(v) = V(G)$). This means that $|E(G)| = k(n-k)$ can hold iff $N(v) = U$ and $N(v) = V(G) \setminus U$, that is, G is a complete bipartite graph with parts of sizes k and $n-k$. This completes the proof. $\square$

From Theorem 4.5 we obtain the following corollary.

Corollary 4.6 *For $r \in \mathbb{N}$ and $k \in \{1, \ldots, \lfloor r/2 \rfloor\}$, we have $d_k(r) = k(n-k)$.*

Proof. Consider a point $u = e_{i_1} + \cdots + e_{i_k}$ of $P_{\mathcal{F}}$ (note that some indices might coincide). As noted before, a neighbor v of u in P must have the form $v = u + \alpha(e_i - e_j)$ for some $\alpha \in \mathbb{N}$, and $i \in [r]$ and $j \in \{i_1, \ldots, i_k\}$. Since each directed edge $(i, j) \in V(\vec{G}(u))$ has its head in $\{i_1, \ldots, i_k\}$, of cardinality at most k, the underlying graph $G(u)$ has a vertex cover of size at most k. Therefore $G(u)$ is a k-tr graph and hence by Theorem 4.5 at most $k(r-k)$ edges.

In the proof of Lemma 4.1 an example of $P_{\mathcal{F}}$ with $|\mathcal{F}| \leq k$ and a vertex of degree $k(n-k)$ was given. This completes the argument. $\square$

5 Minkowski sum of three simplices

In this section we will investigate the polytope $P(3)$. Let $\mathcal{H} := \mathcal{H}(3) = (\{1,2,4,5\}, \{1,2,3,6\}, \{1,3,4,7\})$. Henceforth we will drop the (3). Then $N_{\mathcal{H}}(1) = \{1,2,3\}, N_{\mathcal{H}}(2) = \{1,2\}, N_{\mathcal{H}}(3) = \{2,3\}, N_{\mathcal{H}}(4) = \{1,3\}, N_{\mathcal{H}}(5) = \{1\}, N_{\mathcal{H}}(6) = \{2\}, N_{\mathcal{H}}(7) = \{3\}$, so all of the nonempty subsets of [3] are represented. The case of $k = |\mathcal{F}| = 3$ is the first interesting case for the mere reason that the polytope $P(3)$ does not have $2^{k(k-1)} = 64$ vertices, as was the case for $k = 2$, where the rhombus $P(2)$ had precisely $2^{k(k-1)} = 4$ vertices.

Example: the point $A = (0,1,1,1,0,0,0)$ in $P(3)$ is not a vertex, because $A = (B+C+D)/3$, where $B = (0,2,1,0,0,0,0)$, $C = (0,0,2,1,0,0,0)$ and $D = (0,1,0,2,0,0,0)$ and all the points B, C and D are points in the polytope $P(3)$.

Lemma 5.1 *The polytope $P(3)$ has 41 vertices in $\mathbb{R}^7$ given by the column vectors (without the last entry) in the following 7×10, 7×21 and 7×10 matrices. The last entry in each column is the degree of the vertex.*

$$\begin{array}{cccccccccc}
3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 \\
0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6
\end{array}$$

$$\begin{array}{ccccccccccccccccccccc}
2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
6 & 6 & 6 & 6 & 8 & 6 & 8 & 6 & 6 & 6 & 8 & 6 & 6 & 8 & 6 & 6 & 6 & 6 & 6 & 8 & 8
\end{array}$$

$$\begin{array}{cccccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
7 & 8 & 8 & 7 & 8 & 8 & 7 & 8 & 8 & 9
\end{array}$$

These computations were verified using the computer program POLYMAKE [5]. Using POLYMAKE, we determined that the polytope $P(4)$ had vertices of all degrees in the set $\{14, 15, \ldots, 28\}$ except for $\{16, 23, 26, 27\}$.

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